MATH 2040A Lecture 5 (Sep 22, 2016)

§ Computing eigenvalues / eigenvectors

Recall: $A \in M_{n\times n}(\mathbb{F})$ $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

(*) — $\boxed{A\vec{v} = \lambda \vec{v}}$ $\vec{v} \neq 0$

eigenvector (e.vector) eigenvalue (e. value) $\lambda \in \mathbb{F}$

Rearrange (*)

$$A\vec{v} = \lambda\vec{v} = \lambda I \vec{v}$$

$$\Rightarrow (A - \lambda I)\vec{v} = \vec{0}, \quad \vec{v} \neq \vec{0}$$

Note: ① $\vec{v} \in N(A - \lambda I)$

② $\vec{v} \neq \vec{0} \Rightarrow \det(A - \lambda I) = 0$

Prop: Given $A \in M_{n\times n}(\mathbb{F})$.

(i) $\lambda \in \mathbb{F}$ is an e.value of A $\Longleftrightarrow$ $\boxed{\det(A - \lambda I) = 0}$ "char. eq$^{\underline{n}}$"

(ii) $\vec{0} \neq \vec{v} \in \mathbb{F}^n$ is an e.vector of A $\Longleftrightarrow$ $\boxed{\vec{0} \neq \vec{v} \in N(A - \lambda I)}$

Find e.values / e.vectors of $A \in M_{n\times n}(\mathbb{F})$:

Step 1: Solve char. eq$^{\underline{n}}$ $\rightsquigarrow$ get λ

Step 2: For each $\lambda \in \mathbb{F}$, find $N(A - \lambda I)$.

E.g.: $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

find e.values / e.vectors of A.

Is A diagonalizable?

i.e. $\exists Q$ s.t. $Q^{-1}AQ$ is diagonal.

Step 1: $\boxed{f(t) = \det(A - tI)}$ "char. polynomial"

$$\det(A - tI) = \det\begin{pmatrix} 1-t & 2 \\ 3 & 2-t \end{pmatrix} = t^2 - 3t - 4.$$

Solve: $t^2 - 3t - 4 = 0 \Rightarrow \underbrace{\lambda_1 = 4, \ \lambda_2 = -1}_{\text{"e.values"}}$.

Step 2: For $\lambda_1 = 4$,

$$N(A - 4I) = N\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\}$$

$\therefore \{\text{e.vectors of } A \text{ with } \lambda_1 = 4\} = \text{span}\left\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right\} \setminus \{\vec{0}\}$

For $\lambda_2 = -1$, $N(A + I) = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$.

Take $Q = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \Rightarrow Q^{-1}AQ = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$

diagonalizable!

Eigenvalues / Eigenvectors for T

$T : V \longrightarrow V$ linear. $\dim_{\mathbb{F}} V = n$.

Fix any basis $\beta \subseteq V$,

$$\begin{tikzcd}
\vec{0} \neq \vec{v} \in V \arrow[r, "T"] \arrow[d, leftrightarrow, "\cong_\beta"'] \arrow[dr, phantom, "\circlearrowright"] & V \ni \lambda \vec{v} \arrow[d, leftrightarrow, "\cong_\beta"] \\
\vec{0} \neq [\vec{v}]_\beta \in \mathbb{F}^n \arrow[r, "L_A"'] & \mathbb{F}^n \ni \lambda [\vec{v}]_\beta
\end{tikzcd}$$

$A = [T]_\beta$

Proof by diagram.

Prop: $\vec{v} \in V$ is an e.vector of T w/. e.value $\lambda \in \mathbb{F}$

$\Longleftrightarrow$ $[\vec{v}]_\beta \in \mathbb{F}^n$ is an e.vector of $[T]_\beta$ w/. same e.value $\lambda \in \mathbb{F}$

$\boxed{\text{Find e.vectors / e.values for } T} \rightsquigarrow \boxed{\text{Find e.vectors / e.values for } A = [T]_\beta}$

for ANY β

Eg. $$\begin{array}{ccccc} T : & M_{2\times2}(\mathbb{R}) & \longrightarrow & M_{2\times2}(\mathbb{R}) & \text{linear.} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\ & A & \longmapsto & A^T & \end{array}$$

Step 0 : Write down

$$M := [T]_\beta \quad \text{for any basis } \beta$$

Take a std basis for $M_{2\times2}(\mathbb{R})$ $\dim = 4$.

$$\begin{array}{ccccc} \beta = \Big\{ & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Big\} \\ & \downarrow T & \downarrow T & \downarrow T & \downarrow T \\ & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$M = [T]_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Step 2 : Find e.values / e.vectors for M.

$$\det(M - tI) = (t-1)^3(t+1) = 0$$

$\Longrightarrow$ 2 e.values : $\lambda_1 = 1$, $\lambda_2 = -1$

$\lambda_1 = 1$: $N(M - I) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

For T,

$\lambda_1 = 1$ e.vectors $= \text{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \setminus \vec{0}$

$T\vec{v} = \vec{v}$ $\quad A^T = A$ symmetric matrix

$\lambda_2 = -1$: $N(M + I) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}$

For T,

$\lambda_2 = -1$ e.vectors $= \text{span}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \setminus \vec{0}$

$T\vec{v} = -\vec{v}$ $\quad A^T = -A$ skew-symmetric matrix.